A PROBLEM OF L. V. OVSYANNIKOV

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In 1958 L. V. Ovsyannikov posed to me the problem of studying the first boundary-value problem for the equation $\Delta u = u^2$. This problem arose in a study of near-sonic gas flows [1]. The first results were obtained in 1960-1961 [2-4]. In 1979 a method of separation was proposed in [5] and later developed in [6] for studying nonlinear boundary-value problems in general.

In this paper this method is employed to study the problem posed by L. V. Ovsyannikov. We proposed this problem in the past as one of the applications of the method of separation. The results obtained by this method overlap previous results found by other methods. The result that the solution of a special boundary-value problem for the equation $\Delta u = u^2$ is unique is presented separately. This theorem follows both from the results of Keedy in the exposition of Dancer [7] and the work of Cidas, Ni Wei-Ming, and Nirenberg [8].

1. Derivation of the Equations. The study of three-dimensional near-sonic flows of an ideal polytropic gas leads, as Ovsyannikov showed [1], to an equation of the type

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial z^2}(u-1)^2$$

(u is the reduced projection of the flow velocity on the z axis). The exact solutions of this equation are:

$$u(x, y, z) = 1 + u_0(x, y) + u_1(x, y)z + (1/12)u_2(x, y)z^2.$$

Here the functions $u_0(x, y)$, $u_1(x, y)$, and $u_2(x, y)$ are found from the equations

$$\Delta u_0 - \frac{1}{3}u_2u_0 = 2u_1^2; \tag{1.1}$$

$$\Delta u_1 - u_2u_1 = 0; \tag{1.2}$$

$$\Delta u_2 - u_2^2 = 0. \tag{1.3}$$

Thus the determining equation is Eq. (1.3).

In the case of near-sonic flows adjacent to the sonic plane $z = 0$ on which $u \equiv 1$ we have $u_0(x, y) \equiv u_1(x, y) \equiv 0$, and then the system (1.1)-(1.3) reduces to the single equation (1.3). We note that the positive solution of this equation corresponds to supersonic flow while the negative solution corresponds to subsonic flow.

The equation $\Delta u = u^2$ also arises in the study of some other physical processes. This equation is also of definite interest from the viewpoint of the theory of nonlinear equations as an equation with a leading operator that is even and nonlinear.

2. Formulation of the Problem. Let Ω be a bounded region in $\mathbf{R}^n$, $n \le 5$, with a smooth boundary $\partial\Omega$.

We shall study in the region Ω the boundary-value problem

$$\Delta\Phi + \Phi^2 = 0 \text{ in } \Omega, \quad \Phi = h(x) \text{ on } \partial\Omega \text{ with } h \in W_2^{1/2}(\partial\Omega). \tag{2.1}$$

We note that the equation under study is equivalent by virtue of the substitution $\Phi \to \Phi_1 = -\Phi$ to the equation $\Delta\Phi_1 = \Phi_1{}^2$.

Let H be a harmonic function from $W_2{}^1(\Omega)$ such that $\Delta H = 0$ in Ω and $H = h(x)$ on $\partial\Omega$. Then the starting boundary-value problem is equivalent ($\Phi = u + H$) to the following problem:

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$$\Delta u + (u + H(x))^2 = 0 \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

This problem corresponds to the functional

$$f(u) = \int_\Omega |Du|^2\,dx + \int_\Omega \left[-\frac{2}{3}(u+H)^3 + \frac{2}{3}H^3\right]dx$$

in the Sobolev space $\overset{\circ}{W}{}_2^1(\Omega)$ with the norm $\|u\| = \left(\int_\Omega |Du|^2\,dx\right)^{1/2}$. Then u is the critical point from $\overset{\circ}{W}{}_2^1(\Omega)$ of the function f, i.e., $f'(u) = 0$ is a solution of the boundary-value problem (2.2), and vice versa.

3. Decomposition by the Method of Spherical Separation. Following the method of spherical separation [5] we represent the solution sought $u(x) \not\equiv 0$ of the boundary-value problem (2.2), i.e., the critical point $u \neq 0$ of the functional f, in the form $u = tv$, where $t \in R$ and $v \in S = \{w \in \overset{\circ}{W}{}_2^1(\Omega) \mid \|w\| = 1\}$. Then

$$f(tv) = t^2 + \int_\Omega \left[-\frac{2}{3}(tv+H)^3 + \frac{2}{3}H^3\right]dx$$

$$\left(v \in \overset{\circ}{W}{}_2^1(\Omega) \text{ c } \int_\Omega |Dv|^2\,dx = 1\right).$$

The equation $f'(u) = 0$ with $u \neq 0$ is equivalent, by virtue of the method of separation [5], to the system

$$f_t \equiv 2t - 2\int_\Omega (tv+H)^2 v\,dx = 0; \tag{3.1}$$

$$f_v \equiv -2(tv+H)^2 t = -\lambda\Delta v. \tag{3.2}$$

Here λ is the Lagrange multiplier of the variational problem for the conditional critical point $v \in S$ of the functional $f(tv)$.

From the scalar equation (3.1) for t we find

$$t_i(v) = \left\{1 - 2\int_\Omega Hv^2dx + \nu_i\left[\left(1 - 2\int_\Omega Hv^2dx\right)^2 - \right.\right.$$
$$\left.\left. - 4\int_\Omega H^2v\,dx\int_\Omega v^3dx\right]^{1/2}\right\}\left(2\int_\Omega v^3dx\right)^{-1}$$

$$\text{with } \nu_i = \begin{cases} -1 & \text{for } i = 1, \\ 1 & \text{for } i = 2 \end{cases}$$

and the functionals $F_i(v) = f(t_i(v)v)$ $(i = 1, 2)$ with $\int_\Omega v^3dx \neq 0$.

Then Eq. (3.2) is equivalent to the condition for the existence of critical points of the functionals F_1 and F_2 on the unit sphere S in the Sobolev space $\overset{\circ}{W}{}_2^1(\Omega)$.

Analysis shows that the functional F_1 is defined for all $w \in B_1 = \{w \in \overset{\circ}{W}{}_2^1(\Omega) \mid \|w\| \leq 1\}$, while F_2 is defined for $w \in B_1 \setminus \left\{w \in \overset{\circ}{W}{}_2^1(\Omega) \mid \int_\Omega w^3dx = 0\right\}$.

4. Existence of Real Solutions. It is obvious that the boundary-value problem (2.1) does not have a real solution for arbitrary right sides h from the indicated class. For the real boundary function $h \in W_2^{1/2}(\partial\Omega)$ we assume that the harmonic function H corresponding to it satisfies the equations

$$\sup_{w \in B_1} \int_\Omega Hw^2dx < 1/2; \tag{4.1}$$

$$\inf_{w \in B_1} \left\{\left(1 - 2\int_\Omega Hw^2dx\right)^2 - 4\int_\Omega H^2w\,dx\int_\Omega w^3dx\right\} > 0. \tag{4.2}$$

Under the assumptions made we have

$$\sup_{w\in B_1} F_2(w) = \sup_{v\in S} F_2(v) = +\infty,$$
$$\inf_{w\in B_1} F_2(w) > -\infty, \quad \inf_{w\in B_1} F_1(w) > -\infty.$$

We note that for H(x) = 0 almost everywhere in Ω one solution of the problem (2.1) is the trivial solution. For this reason it is proposed below that

$$\|h\|_{W_2^{1/2}(\partial\Omega)} \neq 0. \tag{4.3}$$

In this case $\inf_{w\in B_1} F_1(w) < 0$. For the functionals F_1 and F_2 in the unit sphere B_1 of the space $\overset{\circ}{W}_2{}^1(\Omega)$ there exist corresponding minimum points w_1 and w_2. For the functional F_1 at the point $w_1 \in B_1$ we have $F_1(w_1) = \min_{w\in B_1} F_1(w) < 0$. From the representation for $t_1(w)$ and $F_1(w)$ we find that $w_1 \neq 0$ and $t_1(w) \neq 0$.

We further establish that at the point w_1 the functionals t_1 and F_1 are differentiable. Then the structure of the functional F_1 determined by the method of spherical separation [$F_1(w) = f(t_1(w)w)$, where $t_1(w)$ satisfies the equation $t - \int_\Omega (tw + H)^2 w\,dx = 0$], shows that the minimum point $w_1 \in B_1$ belongs to S and is therefore a conditional critical point of the functional F_1 on the unit sphere S.

It follows from the method of separation [5] that $u_1 = t_1(w_1)w_1$ is the real critical point of the functional F_1, i.e., the real nontrivial solution of the problem (2.1) under the above-indicated conditions (4.1)-(4.3) on the real function h from the corresponding class.

We shall now examine the functional F_2 under the conditions (4.1) and (4.2) for which there exists a point $w_2 \in B_1$ corresponding to a finite minimum in the unit sphere B_1. At this point $w_2 \neq 0$ we find that the functional t_1 is differentiable and $t_2(w_2) \neq 0$. Then the structure of the functional F_2 shows that this minimum point lies on the unit sphere S.

It then follows from the method of separation [5] that $u_2 = t_2(w_2)w_2$ is the real solution of the boundary-value problem (2.1) with the conditions (4.1) and (4.2).

Summarizing the results obtained and taking into account the trivial solution of the problem (2.1) (with h = 0) we obtain the following result.

THEOREM 4.1. Let the real function h of the boundary-value problem (2.1) belong to the space $W_2{}^{1/2}(\partial\Omega)$ and satisfy conditions (4.1) and (4.2). Then the boundary-value problem (2.1) has different real solutions u_1 and u_2.

The fact that $u_1(x) \not\equiv u_2(x)$ follows from the inequality $t_1(w_1)w_1(x) \not\equiv t_2(w_2)w_2(x)$.

5. Absence of a Real Solution. The method of spherical separation applied to problem (2.2) establishes an equivalence between the variational problem

$$f'(u) = 0 \tag{5.1}$$

in the class of nontrivial solutions from $\overset{\circ}{W}_2{}^1(\Omega)$ and the system

$$\langle f'(tv),\ v\rangle = 0; \tag{5.2}$$
$$tf'(tv) = \lambda\Delta v \tag{5.3}$$

with $f'(tv) \equiv -2t\Delta v - 2(tv + H)^2$ in the class of solutions (t, v) from $(\mathbb{R}\backslash\{0\}) \times S$, where $S = \left\{v \in \overset{\circ}{W}{}_2^1(\Omega) \mid \int_\Omega |Dv|^2\,dx = 1\right\}$.

By virtue of Eq. (5.2) the system (5.2) and (5.3), in its turn, is equivalent in the indicated class of solutions to the system

$$\langle f'(tv),\ v\rangle = 0,\ f'(tv) = 0. \tag{5.4}$$

The absence of a solution (t, v) from $(\mathbf{R}\setminus\{0\}) \times S$ of this system leads to the absence of a solution $u \neq 0$ of the starting variational problem (5.1).

Now let ψ be an arbitrary function from $\overset{\circ}{W}_2{}^1(\Omega)$. It then follows from (5.4) that

$$\langle f'(tv),\ v\rangle = 0, \langle f'(tv),\ \psi\rangle = 0. \tag{5.5}$$

For this reason if there exists a corresponding function $\psi \in \overset{\circ}{W}_2{}^1(\Omega)$ for which (5.5) does not have a solution with $t \neq 0$ and $v \in S$, then the starting variational problem (5.1) also does not have a solution $u \neq 0$ from $\overset{\circ}{W}_2{}^1(\Omega)$.

System (5.5) for problem (2.2) has the form

$$t - \int_\Omega (tv + H)^2 v dx = 0, \quad - t \int_\Omega v \Delta\psi dx - \int_\Omega (tv + H)^2 \psi dx = 0.$$

Here ψ is an arbitrary function from $\overset{\circ}{W}_2{}^1(\Omega)$. The first equation of this system can obviously be derived from the second equation by setting $\psi = v$.

We shall thus study the second scalar equation for t. It obviously does not have a real solution, if there exists a function $\psi \in \overset{\circ}{W}_2{}^1(\Omega)$ such that

$$\left(\int_\Omega (\Delta\psi + 2H\psi) v dx\right)^2 < 4 \int_\Omega H^2\psi dx \int_\Omega \psi v^2 dx \quad \forall v \in S. \tag{5.6}$$

On the other hand, for $\psi(x) \geq 0$ in Ω we have

$$\left(\int_\Omega (\Delta\psi + 2H\psi) v dx\right)^2 = \left(\int_\Omega \frac{\Delta\psi + 2H\psi}{\sqrt{\psi}} \sqrt{\psi}\, v dx\right)^2 \leqslant \int_\Omega \frac{(\Delta\psi + 2H\psi)^2}{\psi} dx \int_\Omega \psi v^2 dx.$$

Inequality (5.6) will hold if there exists a function $\psi \geq 0$ from $\overset{\circ}{W}_2{}^1(\Omega)$ such that

$$\int_\Omega \frac{(\Delta\psi + 2H\psi)^2}{\psi} dx < 4 \int_\Omega H^2 \psi dx.$$

Thus we obtain the following indication that the problem (2.1) does not have a real solution $\Phi \in W_2{}^1(\Omega)$.

Let there exist a function $\psi \geq 0$ from $\overset{\circ}{W}_2{}^1(\Omega)$ such that

$$\int_\Omega \left[\frac{(\Delta\psi)^2}{\psi} + 4H\Delta\psi\right] dx < 0. \tag{5.7}$$

Then the boundary-value problem (2.1) does not have a real solution in the space $W_2{}^1(\Omega)$.

Choosing now the corresponding concrete functions $\psi \geq 0$ from $\overset{\circ}{W}_2{}^1(\Omega)$ we obtain specific indications that the boundary-value problem (2.1) in space $W_2{}^1(\Omega)$ does not have a real solution. We note that the indication (5.7), unlike the traditional indications for the absence of a solution for quasilinear, second-order, elliptical equations is not a point condition, but rather an integral one.

We shall illustrate this for an example. Consider the boundary-value problem (2.1) where Ω is a unit circle in the plane $\mathbf{R}^2$ centered at the origin of coordinates and the boundary function h in polar coordinates equals $A\cos\theta$:

$$\begin{gathered} \Delta\Phi + \Phi^2 = 0 \text{ in } \Omega = \{(x,\ y) \in \mathbf{R}^2 | r^2 = x^2 + y^2 < 1\}, \\ \Phi = A\cos\theta \text{ at } r = 1. \end{gathered} \tag{5.8}$$

Here A is an arbitrary real parameter. This example was chosen for two reasons. First, this problem is presented without analysis in a number of books (see, for example, [9]). Second, and most important, the traditional indications for the nonexistence of a real solution cannot be applied to it, since the mean boundary values vanish: $\int_0^{2\pi} A\cos\theta d\theta = 0$.

Inequality (5.7) for problem (5.8) assumes the form

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left[\frac{(\Delta\psi)^2}{\psi}+4Ar\cos\theta\Delta\psi\right]rdrd\theta<0. \tag{5.9}$$

We now choose as the function ψ the solution of the following problem with the parameter τ: $\Delta\psi = -(\tau + r\cos\theta)(1 - r^2)$ for $r < 1$ and $\psi = 0$ at $r = 1$. This solution can also be written out explicitly and for $\tau \geq 1/3$ the function $\psi \geq 0$.

We substitute the function ψ, depending on $\tau \geq 1/3$, into the inequality (5.9). We then obtain a parametric inequality with $\tau \geq 1/3$ for A, and we find from it an estimate for $|A|$ at $\tau = (1 + \sqrt{5}/2)/3$, $|A| > 20.65$.

Thus, if A satisfies this inequality, then boundary-value problem (5.8) does not have a real solution in the space $W_2{}^1(\Omega)$.

6. Uniqueness of the Nontrivial Solution. Consider in the circle $\Omega = \{(x, y) \in \mathbf{R}^2 | r^2 = x^2 + y^2 < 1\}$ the boundary-value problem

$$\Delta u = u^2 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{6.1}$$

which has the trivial solution $u_1(x) \equiv 0$ and a nontrivial solution $u_2(x) < 0$ in Ω.

Ovsyannikov, in 1959, posed the question of the uniqueness of this nontrivial solution, which, by virtue of what was said in Sec. 1, corresponds to the question of whether or not the regime of near-sonic flow of a gas is unique in the indicated class of solutions.

The possibility of answering this question appeared after the publication of [7, 8].

Keedy's Theorem [7]. Let f be a real nonnegative analytic function. Then any solution of the class $C^2(\bar{\Omega})$ of the problem

$$-\Delta u = f(u) \text{ in } \Omega = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 < 1\},\ u = 0 \text{ on } \partial\Omega$$

is radially symmetric.

Cidas–Ni–Nirenberg Theorem [9]. Let f: $\mathbf{R} \to \mathbf{R}$ be a function of Class C^1. Then any positive solution of the class $C^2(\bar{\Omega})$ of the problem

$$-\Delta u = f(u) \text{ in } \Omega = \{x \in \mathbf{R}^n | |x| < R\},\ u = 0 \text{ on } \partial\Omega$$

is radially symmetric.

Thus, by virtue of either of these theorems the nontrivial solution of problem (6.1) is radially symmetric. Indeed, after the substitution $u \to u_1 = -u$ either theorem is applicable to the boundary-value problem

$$-\Delta u_1 = u_1^2 \text{ in } \Omega,\quad u_1 = 0 \text{ on } \partial\Omega.$$

It remains to verify that the nontrivial solution u(r) of the class $C^2(\bar{\Omega})$ of the boundary-value problem for the ordinary differential equation $\Delta u(r) = u^2(r)$, $u(1) = 0$ is unique. For this we shall employ some arguments from [10].

Assume the converse. Then there exist at least two nontrivial solutions $u_1(r)$ and $u_2(r)$, $u_1(r) \not\equiv u_2(r)$ from the class C^2. From the general theory of quasilinear elliptical equations it follows that the functions $u_1(r)$ and $u_2(r)$ are analytic for $r < 1$. Let $u_1(0) = c_1$ and $u_2(0) = c_2$ and, for definiteness, $c_2 \geq c_1$. We note that $c_1 < 0$ and $c_2 < 0$.

Then u_1 and u_2 are analytical solutions of the Cauchy problem

$$u'' + (1/r)u' = u^2,\ 0 < r < 1,\ u(0) = c,\ u'(0) = 0 \tag{6.2}$$

with $c = c_1$ and $c = c_2$, respectively.

If $c_1 = c_2$, then by virtue of the fact that the solution of the problem (6.2) in the class of analytic functions is unique we obtain $u_1(r) \equiv u_2(r)$ for $r \leq 1$.

In the case $c_2 > c_1$ we shall study the function $v(r) = k^2 u_1(kr)$ with $k = \sqrt{c_2}/c_1 < 1$. Then this function is a solution of problem (6.2) with $c = c_2$, and since its solution is unique in the class of analytic functions we have $u_2(r) = k^2u_1(kr)$ for $r < 1$. Setting in this identity the limiting value $r = 1$, we find that $u_2(1) = k^2u_1(k) < 0$, which contradicts the boundary condition for the solution u_2.

We have thus established that the nontrivial solution of the boundary-value problem (6.1) in the circle $\Omega \subset R^2$ is unique.

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CONSERVATION LAWS, INVARIANCE, AND THE EQUATIONS OF GAS DYNAMICS

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In a large number of papers by L. V. Ovsyannikov, his students, and followers, an analysis was made of group properties of many equations of mathematical physics and it was shown that a knowledge of group properties of the equations is useful for their classification and for obtaining particular solutions (see, for example, [1-3]). An inverse formulation of the problem is also possible: from a given group, sometimes with an additional assumption concerning the transformation law for the desired quantities, to seek the class of differential equations invariant with respect to this group [4, 5]. A similar problem arose, in effect, at roughly the same time, from the theory of relativity, wherein the physics and mathematics began to search for equations describing the dynamics of some range of phenomena dependent on a knowledge of the laws of invariance. From this standpoint the most fundamental object turns out to be a group, and the dynamic equation, in its way, turns out to be the "differential representation" of this group. And just as there exists a supporting theory of linear representations of groups, there exists, indeed, a theory of "differential representations." In this regard, it is necessary to turn our attention to the importance of rational structural limitations of the class of equations being sought. The fundamental equations of mechanics and theoretical physics possess a definite structure. They are usually quasilinear and admit a complete set of conservation laws (see Sec. 1), and, consequently, have a symmetric structure [6-12]. A second simple, but useful observation, consists in the fact that the quantities being sought have, as a rule, a specific tensor type (scalars, vectors, etc.) with respect to suitable transformations. This holds even for the basic conservation laws (mass, momentum, et al.). Only in quantum mechanics do quantities of another

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